Lab 5

Limiting Behaviour of Age-structured Populations

Eigenvalues and eigenvectors

Marine Modelling February 04, 2018

Katja Fennel
Oceanography
Dalhousie University

Recall

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The eigenvector corresponding to the dominant eigenvalue (v_1) gives us the relative age distribution for large values of time. In other words, the age distribution of the population will approach some scalar multiple of that eigenvector with time.

Suppose we have three age classes, females in 2nd and 3rd age class produce 4 and 3 female offspring, and suppose that 50% and 25% of females live to second and third age class, respectively.

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Suppose the initial age distribution is:

$$\underline{x}_0 = \begin{pmatrix} 10 \\ 10 \\ 10 \end{pmatrix}$$

Enter both

```
>> M =[0 4 3; 0.5 0 0; 0 0.25 0];
>> x0 = [10;10;10];
```

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>> X = zeros(3,11);
>> X(:,1) = x0;
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>> X = zeros(3,11);
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```

Now calculate age-distribution over the next 10 years:

```
>> for k=2:11, X(:,k) = M*X(:,k-1); end
```

View the results:

```
>> X
```

```
>> X

 X =

  1.0e+03 *

 Columns 1 through 6

  0.0100   0.0700   0.0275   0.1437   0.0813   0.2978
  0.0100   0.0050   0.0350   0.0138   0.0719   0.0406
  0.0100   0.0025   0.0013   0.0088   0.0034   0.0180

 Columns 7 through 11

  0.2164   0.6261   0.5445   1.3333   1.3238
  0.1489   0.1082   0.3130   0.2722   0.6667
  0.0102   0.0372   0.0271   0.0783   0.0681
```

You noted the scale factor 10^3.

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```
>> format short g
>> X

 X =

 Columns 1 through 6

  10       70      27.5    143.75       81.25      297.81
  10        5        35     13.75      71.875      40.625
  10      2.5      1.25      8.75      3.4375      17.969

  Columns 7 through 11

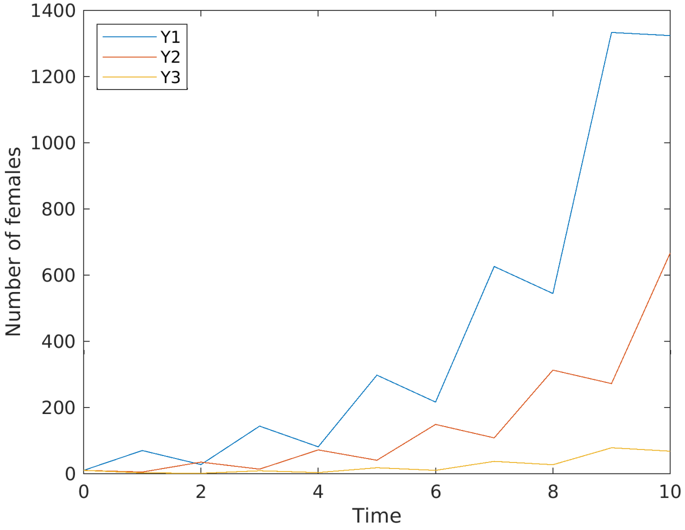
  216.41     626.09     544.49    1333.3     1323.8
  148.91      108.2     313.05    272.25     666.67
  10.156     37.227     27.051    78.262     68.062
```

Plot:

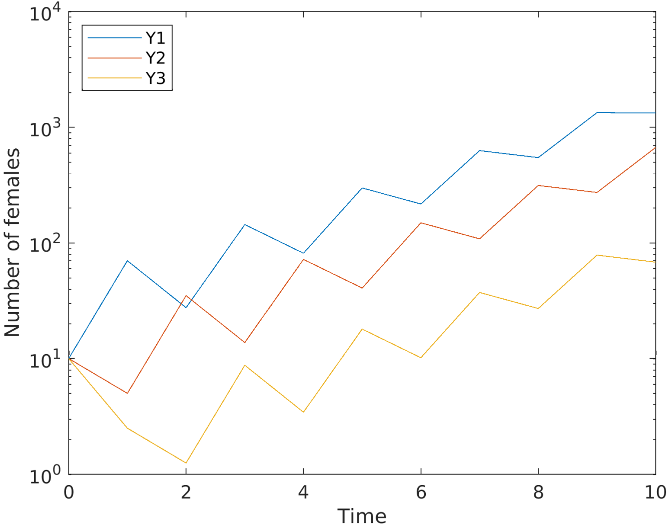
```
>> t=0:10;
>> plot(t,X')
>> xlabel('Time')
>> ylabel('Number of females')
>> legend('Y1','Y2','Y3','Location', 'Best')
```

or plot as semi-log plot:

```
>> figure
>> semilogy(t,X')
>> xlabel('Time')
>> ylabel('Number of females')
>> legend('Y1','Y2','Y3','Location', 'Best')
```



Y1
Y2
Y3
Number of females
Time
0
200
400
600
800
1000
1200
1400
0
2
4
6
8
10


Y1
Y2
Y3
10^4
10^3
10^2
10^1
10^0
0
2
4
6
8
10
Time
Number of females

Now let's calculate the eigenvalues and eigenvectors of M:

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>> [V,D] = eig(M);
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>> [V,D] = eig(M);
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Try `help eig` and read the first two paragraphs. The matrix `V` contains the eigenvectors. The matrix `D` contains the corresponding eigenvalues on it's main diagonal. Let's look at both:

```
>> V
   V =
      0.94737     -0.93201      0.22588
      0.31579        0.356     -0.59137
     0.052632    -0.067989      0.77412

>> D
   D =
          1.5            0            0
            0       -1.309            0
            0            0     -0.19098
```

Let's check that $M = VDV^{-1}$

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```
>> V*D*inv(V)

ans =

   -1.1839e-16             4             3
           0.5   -8.6606e-16   -8.3267e-17
    -6.245e-17          0.25   -5.5511e-17
```

Is this equal to our Leslie matrix M?

Our dominant eigenvalue is 1.5 (`D(1,1)`) and the corresponding eigenvector is the first column of V (i.e. `V(:,1)`)

```
>> v1 = V(:,1)

   v1 =

      0.94737
      0.31579
     0.052632
```

Note that Matlab returns normalized eigenvectors by default:

```
>> v1'*v1

   ans =

       1
```

To get the relative age distribution in the future, we can simply divide each element of `v1` by the sum of all elements in `v1`:

```
>> v1=v1/sum(v1)
   v1 =
         0.72
         0.24
         0.04
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I.e. 72% of the total population will be in age class 1, 24% in age class 2, and 4% in age class 3.

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I.e. 72% of the total population will be in age class 1, 24% in age class 2, and 4% in age class 3.
Let's test this by calculating age distribution after 100 years:

```
>> x100=M^100*x0
   x100 =
   1.1555e+19
   3.8516e+18
   6.4194e+17

>> x = x100/sum(x100)
   x =
         0.72
         0.24
         0.04
```